

# A Modified EMD Algorithm and its Applications

Mayer Humi<sup>1</sup>

<sup>1</sup>Department of Mathematical Sciences,  
Worcester Polytechnic Institute,  
Worcester, MA 01609, USA

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## Abstract

The classical EMD algorithm has been used extensively in the literature to decompose signals that contain nonlinear waves. However when a signal contain two or more frequencies that are close to one another the decomposition might fail. In this paper we propose a new formulation of this algorithm which is based on the zero crossings of the signal and show that it performs well even when the classical algorithm fail. We address also the filtering properties and convergence rate of the new algorithm versus the classical EMD algorithm. These properties are compared then to those of the principal component algorithm (PCA). Finally we apply this algorithm to the detection of gravity waves in the atmosphere.

**Keywords:** Filtering, EMD algorithm

## 1 Introduction

In scientific literature there exist many classical sets of functions which can decompose a signal in terms of "simple" functions. For example Taylor or Fourier expansions are used routinely in scientific and engineering applications.(and many other exist). However in all these expansions the underlying functions are not intrinsic to the signal itself and a precise approximation to the original signal might require a large number of terms. This problem become even more acute when the signal is non-stationary and the process it represents is nonlinear.

To overcome this problem many researchers used in the past the "principal component algorithm" (PCA) to come up with an "adaptive" set of functions which approximate a given signal. A new approach to this problem emerged in the late 1990's when a NASA team has developed the "Empirical Mode Decomposition" algorithm(EMD) which attempt to decompose a signal in terms of it "intrinsic mode functions"(IMF) through a "sifting algorithm". A patent for this algorithm has been issued [1].

The EMD algorithm is based on the following quote [2]: "According to Drazin the first step of data analysis is to examine the data by eye. From this examination, one can immediately identify the different scales directly in two ways: by the time lapse between successive alterations of local maxima and minima and by the time lapse between the successive zero crossings....We have decided to adopt the time lapse between successive extrema as the definition of the time scale for the intrinsic oscillatory mode"

A step by step description of the EMD sifting algorithm is as follows:

1. Let be given a function  $f(t)$  which is sampled at discrete times  $\{t_k, k = 1, \dots, n\}$ .
2. let  $h_0(k) = f(t_k)$ .
3. Identify the max and min of  $h_0(k)$ .
4. Create the cubic spline curve  $M_x$  that connects the maxima points. Do the same for the minima  $M_n$ . This creates an envelope for  $h_0(k)$ .
5. At each time  $t_k$  evaluate the mean  $m_k$  of  $M_x$  and  $M_n$  ( $m_k$  is referred to as the sifting function).
6. Evaluate  $h_1(k) = h_0(k) - m_k$ .
7. If norm of  $\|h_0 - h_1\| < \epsilon$  for some predetermined  $\epsilon$  set the first intrinsic function  $IMF_1 = h_1$  (and stop).
8. if the criteria of (7) are not satisfied set  $h_0(k) = h_1(k)$  and return to (3) ("Sifting process").

The algorithm has been applied successfully in various physical applications. However as has been observed by Flandrin [3] and others the EMD algorithm fails in many cases where the data contains two or more frequencies which are close to each other.

To overcome this difficulty we propose hereby a modification of the EMD algorithm by replacing steps 4 and 5 in the description above by the following:

4. find the midpoints between two consecutive maxima and minima and let  $N_k$  be the values of  $h_0$  at these points.
5. Create the spline curve  $m_k$  that connects the points  $N_k$ .

The essence of this modification is the replacement of the mean which is evaluated by the EMD algorithm as the average of the max-min envelopes by the spline curve of the mid-points between the maxima and minima. This is in line with the observation by Drazin (which was referred to above) that the scales inherent to the data can be educed either from the max-min or its zero crossing. In the algorithm we propose hereby we mimic the "zero-crossings" by the mid-points between the max-min.

It is our objective in this paper to justify this modification of the EMD algorithm through some examples and theoretical work. The plan of the paper is as follows: In Sec. 2 we provides examples of signals composed two or three close frequencies (with and without noise) where the classical EMD algorithm fails but the modified one yields satisfactory results. In Sec. 3 we carry out analytical analysis of the two algorithms which are applied to the same signal. In Sec. 4 we discuss the convergence rate, resolution and

related issues concerning the classical and new "midpoint algorithm" . Sec. 5 address the application of this algorithm to atmospheric data and in Sec. 6 we compare the EMD and PCA algorithms

## 2 Examples and Comparisons

Extensive experimentations were made to test and verify the efficiency of the modified algorithm. We present here the results of one of these tests in which the signal contains three close frequencies. (In our tests we considered also the effects of noise and phase shifts among the different frequencies)

$$f(t) = \frac{1}{3}[\cos(\omega_1 t) + \cos(\omega_2 t) + \cos(\omega_3 t)] \quad (1)$$

where

$$\omega_1 = 12\omega_0, \quad \omega_2 = 10\omega_0, \quad \omega_3 = 8\omega_0, \quad \omega_0 = \frac{\pi}{256}.$$

To apply the EMD algorithm to this signal, we used a discrete representation of it over the interval  $[-2048, 2048]$  by letting  $t_{k+1} - t_k = 1, k = 1, \dots, 4097$ .

The results of the signal decompositions into IMFs and a comparison these IMFs with the frequencies present in the original signal are presented in figures 1 – 5. In all these figures the red lines represent the frequencies in the original signal (or its power spectrum) and the blue lines the corresponding intrinsic mode functions or their power spectrum which were obtained by the midpoint algorithm.

Fig. 1 is a plot of the data for the signal described by (1). Fig. 2 represents the first IMF in the decomposition (versus the leading frequency in the data) while Figs. 3 – 5 depict the spectral density distribution for the first three IMFs versus those related to the original frequencies in the data. It should be observed that although the amplitude of the spectral densities in these plots are different (especially for IMF 3) the maxima of the spectral density in each plot is very close to the original one.

The EMD algorithm is a high pass filter. For the  $n - th$  iteration of the filter its efficiency is measured by the parameter  $\alpha$  which is defined by

$$Y_n = \alpha_n Y_{n-1} + \alpha(X_n - X_{n-1})$$

where  $X_k$  and  $Y_k$  are the input and output of the  $k - th$  iteration. Fig 6 present the value of the parameter  $\alpha$  as a function of the iteration number for first IMF derived from the data of the signal in (1).

## 3 Some Analytical Insights

To obtain analytical insights about the performance of the EMD-midpoint algorithm we considered the following signal

$$f(t) = \frac{1}{2}[\cos(\omega_4 t) + \cos(\omega_5 t)], \quad \omega_4 = \frac{3\pi}{64}, \quad \omega_5 = \frac{\pi}{32}. \quad (1)$$

Since the ratio of the frequencies in this signal is a rational number the signal is actually periodic with period  $p = 128$  (See Fig. 7) and the behavior of the classical versus the mid-point algorithm can be delineated analytically (i.e without discretizations).

On the interval  $[0, p]$  the extrema of the signal are given by  $\frac{df}{dt} = 0$  and therefore it is easy to construct the spline approximation  $S_{max}(t)$ ,  $S_{min}(t)$  to the maximum and minimum points and compute their average. Similarly we can find the midpoints between the maxima and minima and evaluate the corresponding spline approximation  $S_{mid}(t)$  to the signal at these points. after one iteration of the sifting process the "sifted signal" is given respectively by

$$h_{mn}(t) = f(t) - \frac{S_{max}(t) + S_{min}(t)}{2}, \quad (2)$$

and

$$h_{mid}(t) = f(t) - S_{mid}(t). \quad (3)$$

The efficiency of the two algorithm can be deduced by projecting these new signals on the Fourier components of the original signal. To this end we compute

$$a_{mn} = \int_0^p h_{mn}(t) \cos(\omega_4 t) dt, \quad b_{mn} = \int_0^p h_{mn}(t) \sin(\omega_4 t) dt. \quad (4)$$

$$c_{mn} = \int_0^p h_{mn}(t) \cos(\omega_5 t) dt, \quad d_{mn} = \int_0^p h_{mn}(t) \sin(\omega_5 t) dt. \quad (5)$$

and

$$a_{mid} = \int_0^p h_{mid}(t) \cos(\omega_4 t) dt, \quad b_{mid} = \int_0^p h_{mid}(t) \sin(\omega_5 t) dt. \quad (6)$$

$$c_{mid} = \int_0^p h_{mid}(t) \cos(\omega_4 t) dt, \quad d_{mid} = \int_0^p h_{mid}(t) \sin(\omega_5 t) dt. \quad (7)$$

The amplitude of the Fourier components of the two frequencies in the classical EMD algorithm is

$$A_{mn} = \sqrt{a_{mn}^2 + b_{mn}^2}, \quad B_{mn} = \sqrt{c_{mn}^2 + d_{mn}^2}. \quad (8)$$

Similarly for the mid-point algorithm we

$$A_{mid} = \sqrt{a_{mid}^2 + b_{mid}^2}, \quad B_{mid} = \sqrt{c_{mid}^2 + d_{mid}^2}. \quad (9)$$

The objective of the sifting process is to eliminate one of the Fourier components in favor of the other. As a result the first IMF will contains, upon convergence, only one of the Fourier components in the original signal. Therefore the efficiency of the two algorithm can be inferred by comparing  $A_{mn}$  versus  $B_{mn}$  and  $A_{mid}$  versus  $B_{mid}$ . Computing the integrals that appear in eqs.(4)-(7) we obtain

$$A_{mn} = 31.63346911, \quad B_{mn} = 29.70292046, \quad (10)$$

$$A_{mid} = 34.19647843, \quad B_{mid} = 20.81145369. \quad (11)$$

These results show that after one iteration the classical EMD did not separate the two frequencies effectively. On the other hand the mid-point algorithm performed well.

## 4 Convergence Rates

To compare the convergence rates of the classical versus the midpoint algorithm we considered three cases all of which were composed of two frequencies. In the first case the two frequencies were well separated. In the second case the two frequencies were close while in the third case they were almost "overlapping". In all cases the signal was given by

$$f(t) = \frac{1}{2}(\cos \omega_1 t + \cos \omega_2 t)$$

This signal was discretized on the interval  $[-2048, 2048]$  with  $\Delta t = 1$ .

For the first case the two frequencies were

$$\omega_1 = 12\omega, \quad \omega_2 = 8\omega, \quad \omega = \frac{\pi}{256}.$$

As can be expected both the classical and midpoint algorithm were able to discern the individual frequencies through the sifting algorithm. However it took the classical algorithm 59 iterations to converge to the first IMF. On the other hand the midpoint algorithm converged in only 7 iterations (using the same convergence criteria). We wish to point out also that the midpoint algorithm has a lower computational cost than the classical algorithm. It requires in each iteration the computation of only one spline interpolating polynomial. On the other hand the classical algorithm requires two such polynomials, one for the maximum points and one for the minimum points.

For the second test the frequencies were

$$\omega_1 = \frac{\pi}{24} + \frac{\pi}{288}, \quad \omega_2 = \frac{\pi}{24} - \frac{\pi}{288}$$

that is the difference between the two frequencies is  $\frac{\pi}{144}$ .

In this case the midpoint algorithm was able to separate the two frequencies. Fig 8 and Fig 9 compare the power spectrum of the original frequencies versus those of  $IMF_1$  and  $IMF_2$  which were obtained through this algorithm. Convergence to  $IMF_1$  was obtained in 18 iterations and  $IMF_2$  was obtained by 7 additional iterations.

The classical EMD algorithm did converge to  $IMF_1$  in 45 iterations but the power spectrum of this  $IMF$  deviated significantly from the first frequency in the signal (See Fig 10).  $IMF_2$  failed (completely) to detect correctly the second frequency.

In third case the frequencies were

$$\omega_1 = \frac{\pi}{24} + \frac{\pi}{1000}, \quad \omega_2 = \frac{\pi}{24} - \frac{\pi}{1000}.$$

In this case the classical algorithm was unable to separate the two frequencies i.e  $IMF_1$  contained both frequencies (See Fig 11). The midpoint algorithm did somewhat better but the resolution was not complete (See Fig 12). Moreover the sifting process in both cases led to the creation of "ghost frequencies" which were not present in the original signal.

At this juncture one might wonder if a "hybrid algorithm" whereby the sifting function is the average (or some similar combination) of those obtained by the classical and midpoint algorithms might outperform the separate algorithms (in spite of the obvious additional computational cost). However our experimentations with such algorithm did not yield the desired results (i.e. the convergence rate and resolution did not improve).

## 5 Applications to Atmospheric Data

There have been recent interest in the observation and properties of gravity waves which are generated when wind is blowing over terrain. In part this interest stems from the fact that these waves carry energy and accurate measure of this data is needed to improve the performance of numerical weather prediction models.

As part of this scientific campaign the USAF flew several balloons that collected information about the pressure and temperature as a function of height. The temperature data collected by one of these balloons is presented in Fig. 13 [6]. To analyze this signal we detrended first it by subtracting its mean from the data. When the mid-point EMD algorithm was applied to this detrended-signal the first IMF extracted the experimental noise from while the second and third IMFs educed clearly the gravity waves (the second IMF is depicted in Fig. 14). On the other hand the classical EMD algorithm failed to educe these waves from the detrended-signal.

Subtracting the gravity waves that were detected by the mid-point algorithm from the detrended-signal we obtain the "turbulent residuals" whose spectrum is shown in Fig 15. The slope of this signal in the "inertial frequency range" is  $-2.7$  which corresponds well with the fact that the flow in stratosphere is "quasi two-dimensional" [7-9].

## 6 EMD or PCA- A Comparison

Before the emergence of the EMD algorithm an adaptive data analysis was provided by the "Principal Component Algorithm"(PCA) which is referred to also as the "Karahunan-Loeve (K-L) decomposition algorithm". (For a review see [10]) Here we shall give only a brief overview of this algorithm within in the geophysical context.

Let a signal be represented by a a time series  $X$  (of length  $N$ ) of some variable. We first determine a time delay  $\Delta$  for which the points in the series are decorrelated. Using  $\Delta$  we create  $n$  copies of the original series

$$X(k), X(d + \Delta), \dots, X(k + (n - 1)\Delta).$$

(To create these one uses either periodicity or choose to consider shorter time-series). Then one computes

the auto-covariance matrix  $R = (R_{ij})$

$$R_{ij} = \sum_{k=1}^N X(k + i\Delta)X(k + j\Delta). \quad (1)$$

Let  $\lambda_0 > \lambda_1, \dots, > \lambda_{n-1}$  be the eigenvalues of  $R$  with their corresponding eigenvectors

$$\phi^i = (\phi_0^i, \dots, \phi_{n-1}^i), \quad i = 0, \dots, n-1.$$

The original time series  $X$  can be reconstructed then as

$$X(j) = \sum_{k=0}^{n-1} a_k(j) \phi_0^k \quad (2)$$

where

$$a_k(j) = \frac{1}{n} \sum_{i=0}^{n-1} X(j + i\Delta) \phi_i^k. \quad (3)$$

The essence of the PCA is based on the recognition that if a large spectral gap exists after the first  $m_1$  eigenvalues of  $R$  then one can reconstruct the mean flow (or the large component ( of the data by using only the first  $m_1$  eigenfunctions in (2). A recent refinement of this procedure due to Ghil et al ([10]) is that the data corresponding to eigenvalues between  $m_1 + 1$  and up to the point  $m_2$  where they start to form a “continuum” represent waves. The location of  $m_2$  can be ascertained further by applying the tests devised by Axford [11] and Dewan [7].

Thus the original data can be decomposed into mean flow, waves and residuals (i.e. data corresponding to eigenvalues  $m_2 + 1, \dots, n-1$  which we wish to interpret at least partly as turbulent residuals).

The crucial step in this algorithm is the determination of the points  $m_1$  and  $m_2$  whose position has to be ascertained by additional tests whose results might be equivocal.

We applied this algorithm to the geophysical data described in Sec. 5.1 with  $\Delta = 96$  and computed the resulting spectrum of the correlation matrix  $R$ . This spectrum is depicted in Fig. 16 . Based on this spectrum we choose  $m_1 = 3$  and  $m_2 = 11$  we obtain the corresponding wave component of the signal that is shown in Fig. 17.

We conclude that while the PCA algorithm provides an alternative to the EMD algorithm the determination of the cutoff points is murky in many cases. However it will be advantageous if one apply the two algorithms in tandem in order to obtain a clear cut confirmation of the results.

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